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A numerical scheme for the Maxwell equations in the quasi-static regime with a non-local source

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ABSTRACT

In this paper we study a non-linear evolution equation, based on quasi-static electromagnetic fields, with a non-local field-dependent source. This model occurs in transformer driven active magnetic shielding. We present a numerical scheme for both time and space discretization and prove convergence of this scheme. We also derive the corresponding error estimates.

1. Introduction

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$ and an outward unit normal vector $\mathbf{n}$. We consider the following quasi-static Maxwell system with a non-local field-dependent source term,

$$\begin{cases} \sigma\partial_t\mathbf{A} + \nabla\times(\nu\nabla\times\mathbf{A}) = \mathbf{J}(\mathbf{A}, \nabla\times\mathbf{A}), & \text{in } [0,T]\times\Omega, \\ \mathbf{n}\times\mathbf{A} = \mathbf{0}, & \text{on } [0,T]\times\partial\Omega, \\ \mathbf{A}(0) = \mathbf{A}_0, & \text{in } \Omega, \end{cases} \tag{1}$$

where σ and ν denote the conductivity and inverse permeability of the domain. We consider linear anisotropic materials such that σ, ν are functions in the space $(L^\infty(\Omega))^{3\times3}$, symmetric and uniformly positive, i.e., there exist positive constants σ_m, σ_M (and analogous constants ν_m, ν_M for ν) such that for every $\mathbf{x}$ in Ω and every $\boldsymbol{\xi}\in\mathbb{R}^3$,

$$0 < \sigma_m|\boldsymbol{\xi}|^2 \le \sum_{i,j=1}^{3}\sigma_{ij}(\mathbf{x})\xi_i\xi_j \le \sigma_M|\boldsymbol{\xi}|^2. \tag{2}$$

Notation. The standard scalar product in $(L^2(\Omega))^3$ will be denoted by $(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx$ and the norm in this space is $\|\mathbf{u}\| = \sqrt{(\mathbf{u},\mathbf{u})}$. The Euclidean norm of a vector $\boldsymbol{\xi}$ in $\mathbb{R}^3$ is $|\boldsymbol{\xi}|$. By $\|\mathbf{u}\|_{\Omega_0}$ we denote $\sqrt{\int_{\Omega_0}|\mathbf{u}|^2dx}$. C, ε and C_ε are general constants, where ε is arbitrarily small and C_ε arbitrarily large. The same notation will be used for different constants, but the meaning

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will be clear from the context. For ease of notation, we will refer to the source function as $\mathbf{J}(\mathbf{A})$ in the further analysis and omit the second argument $\nabla \times \mathbf{A}$. We will work in the standard Hilbert spaces $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}_0(\mathbf{curl}, \Omega)$, provided with the norm $\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} = \|\mathbf{u}\|^2 + \|\nabla \times \mathbf{u}\|^2$, as well as the spaces $\mathbf{H}^\alpha(\mathbf{curl}, \Omega)$ for interpolation in curl-conforming finite element spaces, defined as

$$\begin{aligned}
&\mathbf{H}(\mathbf{curl}, \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 \mid \nabla \times \mathbf{u} \in (L^2(\Omega))^3\},\\
&\mathbf{H}_0(\mathbf{curl}, \Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \mid \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\},\\
&\mathbf{H}^\alpha(\mathbf{curl}, \Omega) = \{\mathbf{v} \in (H^\alpha(\Omega))^3 \mid \nabla \times \mathbf{v} \in (H^\alpha(\Omega))^3\}.
\end{aligned}$$

Finally we state some useful (in)equalities which can easily be derived and which are essential for the a priori estimates.

$$2\sum_{i=1}^{j} \left(\sigma \left(\mathbf{A}_i - \mathbf{A}_{i-1}\right), \mathbf{A}_i\right) = \left(\sigma \mathbf{A}_j, \mathbf{A}_j\right) - \left(\sigma \mathbf{A}_0, \mathbf{A}_0\right) + \sum_{i=1}^{j} \left(\sigma \left(\mathbf{A}_i - \mathbf{A}_{i-1}\right), \mathbf{A}_i - \mathbf{A}_{i-1}\right), \tag{3}$$

$$ab \le \varepsilon a^2 + C_\varepsilon b^2, \quad a, b \in \mathbb{R},\ \varepsilon > 0. \text{ [Young's inequality]} \tag{4}$$

The source function $\mathbf{J}$ is a Lipschitz continuous function from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega)$, i.e., there exists a constant L such that

$$\|\mathbf{J}(\mathbf{u}_1, \mathbf{v}_1) - \mathbf{J}(\mathbf{u}_2, \mathbf{v}_2)\| \le L \left(\|\mathbf{u}_1 - \mathbf{u}_2\| + \|\mathbf{v}_1 - \mathbf{v}_2\|\right), \tag{5}$$

for all $(\mathbf{u}_1, \mathbf{v}_1)$, $(\mathbf{u}_2, \mathbf{v}_2)$ in $L^2(\Omega) \times L^2(\Omega)$. This general definition of the non-linear source allows us to consider the case where the source depends both on the vector potential $\mathbf{A}$ as on the magnetic induction $\nabla \times \mathbf{A}$.

Remark 1. We define the Lipschitz continuity in a non-local way using $L^2(\Omega)$-norms on both sides of the inequality. This is a more general approach than the standard pointwise definition of Lipschitz continuity for a function from $\mathbf{R}^3 \times \mathbf{R}^3$ to $\mathbf{R}^3$

$$|\mathbf{J}(\mathbf{x}_1, \mathbf{y}_1) - \mathbf{J}(\mathbf{x}_2, \mathbf{y}_2)| \le L \left(|\mathbf{x}_1 - \mathbf{x}_2| + |\mathbf{y}_1 - \mathbf{y}_2|\right), \quad \text{for all } \mathbf{x}_1, \mathbf{x}_2 \text{ in } \mathbb{R}^3,$$

since (5) is also valid for functions $\mathbf{J}(\mathbf{u}, \mathbf{v}) = \mathcal{J}(\|\mathbf{u}\|_{\Omega_0}, \|\mathbf{v}\|_{\Omega_0})$, where $\mathcal{J}$ is pointwise Lipschitz continuous and Ω_0 is a compact subset of Ω.

Initial data $\mathbf{A}_0$ should satisfy

$$\mathbf{A}_0 \in \mathbf{H}(\mathbf{curl}, \Omega). \tag{6}$$

Under additional conditions on $\mathbf{A}_0$ a better convergence rate can be obtained.

In [1] error estimates are presented for a finite element method for the time harmonic Maxwell equations. The estimates are based on Nédélec's first type of elements [2]. For vanishing conductivity, the studied problem is not coercive and a discrete Helmholtz decomposition is applied to analyze the error. The first analysis of the use of Nédélec's elements for the time-dependent Maxwell equations was done in [3]. It is based on a semi-discrete approximation (no time discretization), where the problem of non-coercivity is again solved using a Helmholtz decomposition. The first fully discrete finite element approach for the Maxwell equations was studied in [4]. A backward Euler scheme is combined with Nédélec's second family of finite elements and optimal error estimates are obtained for solutions with sufficient regularity.

The previous results are valid for linear materials and a linear source term. In this article we wish to extend this to a semi-linear electromagnetic problem, occurring in magnetic shielding. We introduce a highly non-linear source term which depends on the value of the magnetic induction in a certain region. In Section 2 we present the variational formulation of our problem and prove uniqueness of the weak solution. Section 3 presents the necessary estimates for the semi-discrete problem, in order to prove convergence of our numerical method in Section 4. In Section 5 we prove that the limit of our sequence of approximations is exactly the weak solution to our problem. Finally, we study the fully discretized system in Section 6 and derive the error estimates. The last section presents some numerical simulations of an inverse shielding problem in electromagnetism.

2. Variational formulation

The variational formulation of (1) reads as

$$(\sigma \partial_t \mathbf{A}, \boldsymbol{\varphi}) + (\nu \nabla \times \mathbf{A}, \nabla \times \boldsymbol{\varphi}) = (\mathbf{J}(\mathbf{A}), \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega). \tag{7}$$

We first prove that if the variational problem has a solution (this is proved in Theorem 7), then this solution is unique.

Theorem 1 (*Uniqueness*). *The variational problem (7) admits at most one solution in* $\mathbf{H}_0(\mathbf{curl}, \Omega)$.

Proof. Suppose $\mathbf{A}_1, \mathbf{A}_2$ both satisfy (7). After subtraction of the identity for $\mathbf{A}_1$ and $\mathbf{A}_2$, taking $\boldsymbol{\varphi} = \mathbf{A}_1 - \mathbf{A}_2$ and time integration, we get from the positivity of σ and ν

$$\frac{1}{2}\sigma_m \|\mathbf{A}_1(t) - \mathbf{A}_2(t)\|^2 + \nu_m \int_0^t \|\nabla \times \mathbf{A}_1 - \nabla \times \mathbf{A}_2\|^2 \mathrm{d}t \le \int_0^t \left(\mathbf{J}(\mathbf{A}_1) - \mathbf{J}(\mathbf{A}_2), \mathbf{A}_1 - \mathbf{A}_2\right) \mathrm{d}t$$

Applying the Cauchy–Schwarz and Young's inequalities, the Lipschitz continuity of **J** results in

$$\|\mathbf{A}_1(t)-\mathbf{A}_2(t)\|^2+\int_0^t\|\nabla\times\mathbf{A}_1-\nabla\times\mathbf{A}_2\|^2\mathrm{d}t\le C_\varepsilon\int_0^t\|\mathbf{A}_1-\mathbf{A}_2\|^2\mathrm{d}t+\varepsilon\int_0^t\|\nabla\times\mathbf{A}_1-\nabla\times\mathbf{A}_2\|^2\mathrm{d}t.$$

If we choose ε small enough, the Gronwall inequality allows us to conclude that $\|\mathbf{A}_1-\mathbf{A}_2\|=0$ for all $t\in[0,T]$. $\square$

We will introduce a backward Euler discretization to solve the problem numerically. The time interval $[0,T]$ is equally divided in n subintervals $[t_{i-1},t_i]$ with timestep $\tau=T/n$, thus $t_i=i\tau$, $0\le i\le n$. The standard notation for the discretized fields is

$$\mathbf{H}_i=\mathbf{H}(t_i),\qquad \delta\mathbf{H}_i=\frac{\mathbf{H}_i-\mathbf{H}_{i-1}}{\tau}.$$

After time discretization we obtain the following recurrent approximation scheme for (7) $(i=1,\dots,n)$

$$(\sigma\delta\mathbf{A}_i,\boldsymbol{\varphi})+(\nu\nabla\times\mathbf{A}_i,\nabla\times\boldsymbol{\varphi})=(\mathbf{J}(\mathbf{A}_{i-1}),\boldsymbol{\varphi}),\quad \forall\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl},\Omega),\tag{8}$$

which is equivalent to

$$\begin{aligned}a(\mathbf{A}_i,\boldsymbol{\varphi}) &:= \left(\sigma\frac{\mathbf{A}_i}{\tau},\boldsymbol{\varphi}\right)+(\nu\nabla\times\mathbf{A}_i,\nabla\times\boldsymbol{\varphi})\\ &= (\mathbf{J}(\mathbf{A}_{i-1}),\boldsymbol{\varphi})+\left(\sigma\frac{\mathbf{A}_{i-1}}{\tau},\boldsymbol{\varphi}\right)=:f(\boldsymbol{\varphi}).\end{aligned}\tag{9}$$

From the properties (2) on σ and ν, it follows that a is a continuous and coercive bilinear form on $\mathbf{H}_0(\mathbf{curl},\Omega)$ and that f is a continuous functional on $\mathbf{H}_0(\mathbf{curl},\Omega)$. Using the Lax–Milgram lemma we immediately obtain the existence and uniqueness of the solution $\mathbf{A}_i\in\mathbf{H}_0(\mathbf{curl},\Omega)$ of (9) for every $i=1,\dots,n$, starting from the initial value $\mathbf{A}_0$.

3. A priori estimates

Based on the time discretized variational formulation, we can deduce some estimates on the fields $\mathbf{A}_i$ and their derivatives.

Lemma 2. *Let $\mathbf{A}_0\in\mathbf{H}(\mathbf{curl},\Omega)$. Then there exists a positive constant C, such that*

$$\max_{j=1,\dots,n}\|\mathbf{A}_j\|^2+\sum_{i=1}^n\tau\|\nabla\times\mathbf{A}_i\|^2+\sum_{i=1}^n\|\mathbf{A}_i-\mathbf{A}_{i-1}\|^2\le C.\tag{10}$$

Proof. We take $\boldsymbol{\varphi}=\tau\mathbf{A}_i$ in (8) and add for $i=1,\dots,j$. Using the identity (3) and the properties of σ and ν, we obtain

$$\|\mathbf{A}_j\|^2+\sum_{i=1}^j\|\mathbf{A}_i-\mathbf{A}_{i-1}\|^2+\sum_{i=1}^j\tau\|\nabla\times\mathbf{A}_i\|^2\le C\|\mathbf{A}_0\|^2+C\tau\sum_{i=1}^j(\mathbf{J}(\mathbf{A}_{i-1}),\mathbf{A}_i).$$

From the Lipschitz continuity (5) of **J**, we get $\|\mathbf{J}(\mathbf{A})\|^2\le C(1+\|\mathbf{A}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)})$. Using Cauchy's and Young's inequalities, the RHS can be bounded as follows

$$\tau\sum_{i=1}^j(\mathbf{J}(\mathbf{A}_{i-1}),\mathbf{A}_i)\le C_\varepsilon\sum_{i=1}^j\tau\|\mathbf{A}_i\|^2+\varepsilon\sum_{i=1}^j\tau\|\mathbf{A}_i\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}+C.$$

Since ε can be chosen arbitrarily small, we obtain

$$\|\mathbf{A}_j\|^2+\sum_{i=1}^j\|\mathbf{A}_i-\mathbf{A}_{i-1}\|^2+\sum_{i=1}^j\tau\|\nabla\times\mathbf{A}_i\|^2\le C+C\sum_{i=1}^j\tau\|\mathbf{A}_i\|^2.$$

From the discrete Gronwall inequality for the non-negative sequence $\|\mathbf{A}_j\|^2$ we obtain the boundedness of the sequence $\|\mathbf{A}_j\|^2$, $j=1,\dots,n$. This proves the lemma. $\square$

Lemma 3. *Suppose $\mathbf{A}_0\in\mathbf{H}(\mathbf{curl},\Omega)$. Then there exists a positive constant C such that*

$$\max_{i=1,\dots,n}\|\nabla\times\mathbf{A}_i\|^2+\sum_{i=1}^n\tau\|\delta\mathbf{A}_i\|^2+\sum_{i=1}^n\|\nabla\times\mathbf{A}_i-\nabla\times\mathbf{A}_{i-1}\|^2\le C.\tag{11}$$

Proof. We set $\boldsymbol{\varphi}=\tau\delta\mathbf{A}_i$ in (8) and add for $i=1,\dots,j$. The right-hand side can be bounded using the Lipschitz continuity (5), Cauchy's and Young's inequalities and Lemma 2. $\square$

Lemma 4. *Suppose $\nu\nabla \times \mathbf{A}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then there exists a constant C, such that*

$$\max_{i=1,\dots,n} \|\delta \mathbf{A}_i\|^2 + \sum_{i=1}^{n} \tau \|\nabla \times \delta \mathbf{A}_i\|^2 + \sum_{i=1}^{n} \|\delta \mathbf{A}_i - \delta \mathbf{A}_{i-1}\|^2 \le C. \tag{12}$$

Proof. We subtract (8) at timestep i from the expression at timestep $i-1$, put $\boldsymbol{\varphi} = \delta \mathbf{A}_i$ and add for $i = 1, \dots, j$. To give meaning to $\delta \mathbf{A}_0$, we consider the weak form (7) at $t = 0$. We obtain the following compatibility condition

$$\delta \mathbf{A}_0 := \sigma^{-1} \mathbf{J}(\mathbf{A}_0) - \sigma^{-1} \nabla \times (\nu \nabla \times \mathbf{A}_0)$$

If $\nu\nabla \times \mathbf{A}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$, we have $\delta \mathbf{A}_0 \in (L^2(\Omega))^3$.

From the Lipschitz continuity of $\mathbf{J}$, Lemmas 2 and 3 and the Gronwall lemma, we obtain the desired result. □

4. Convergence

We prove the existence of the weak solution using Rothe's method [5]. We introduce the piecewise linear vector fields $\mathbf{A}_n$, where n denotes the number of subintervals in $[0, T]$

$$\begin{aligned}
&\mathbf{A}_n(0) = \mathbf{A}_0,\\
&\mathbf{A}_n(t) = \mathbf{A}_{i-1} + (t - t_{i-1})\delta \mathbf{A}_i, \quad t \in (t_{i-1}, t_i],
\end{aligned}$$

and the piecewise constant fields $\overline{\mathbf{A}}_n$ as

$$\begin{aligned}
&\overline{\mathbf{A}}_n(0) = \mathbf{A}_0,\\
&\overline{\mathbf{A}}_n(t) = \mathbf{A}_i, \quad t \in (t_{i-1}, t_i].
\end{aligned}$$

With these fields, we can rewrite (8) as

$$(\sigma \partial_t \mathbf{A}_n(t), \boldsymbol{\varphi}) + (\nu \nabla \times \overline{\mathbf{A}}_n(t), \nabla \times \boldsymbol{\varphi}) = (\mathbf{J}(\overline{\mathbf{A}}_n(t - \tau)), \boldsymbol{\varphi}). \tag{13}$$

The strength of Rothe's method is that we can prove the convergence of the sequences $\mathbf{A}_n$ and $\overline{\mathbf{A}}_n$ to the unique weak solution of (1) as $\tau \to 0$ ($n \to \infty$). We will first prove some basic properties for the sequences $\mathbf{A}_n$ and $\overline{\mathbf{A}}_n$.

Lemma 5. *The sequences $\mathbf{A}_n$ and $\overline{\mathbf{A}}_n$ have the same limit in $L^2((0, T), \mathbf{H}(\mathbf{curl}, \Omega))$.*

Proof. From the definition of the Rothe functions, we immediately get

$$\begin{aligned}
\int_0^T \|\mathbf{A}_n - \overline{\mathbf{A}}_n\|^2 \mathrm{d}t \ &= \ \sum_{i=1}^{n} \|\delta \mathbf{A}_i\|^2 \int_{t_{i-1}}^{t_i} |t - t_{i-1} - \tau|^2 \mathrm{d}t \\
&\le \ \tau^2 \sum_{i=1}^{n} \tau \|\delta \mathbf{A}_i\|^2 \\
&\le \ C\tau^2, \quad (\text{Lemma 3}) \\
\int_0^T \|\nabla \times (\mathbf{A}_n - \overline{\mathbf{A}}_n)\|^2 \mathrm{d}t \ &\le \ \tau^2 \sum_{i=1}^{n} \tau \|\nabla \times \delta \mathbf{A}_i\|^2 \\
&\to \ \le \ C\tau, \quad (\text{Lemma 3}) \\
&\to \ \le \ C\tau^2. \quad (\text{Lemma 4}).
\end{aligned}$$

The additional condition $\nu\nabla \times \mathbf{A}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$ from Lemma 4 results in a faster convergence. □

Theorem 6 (*Convergence*). *Let the assumptions on $\mathbf{A}_0$ in Lemmas 2 and 3 be satisfied. Then there exists an $\mathbf{A}$ in $C([0, T], (L^2(\Omega))^3) \cap L^2([0, T], \mathbf{H}(\mathbf{curl}, \Omega))$, such that $\mathbf{A}_n$ converges to $\mathbf{A}$ in both spaces.*

Proof. Consider two different stepsizes $\tau_r = T/r$ and $\tau_s = T/s$. The corresponding variational formulation (8) can be solved in both cases and we can construct Rothe's fields $\mathbf{A}_r$, $\overline{\mathbf{A}}_r$, $\mathbf{A}_s$ and $\overline{\mathbf{A}}_s$. These fields satisfy (13). We subtract both equations, set $\boldsymbol{\varphi} = \mathbf{A}_r - \mathbf{A}_s$ and integrate in time over $(0, \xi)$. Together with the positivity (2) of σ and ν, we obtain

$$\begin{aligned}
\frac{1}{2}\sigma_m \|\mathbf{A}_r(\xi) - \mathbf{A}_s(\xi)\|^2 + \nu_m \int_0^\xi \|\nabla \times (\mathbf{A}_r - \mathbf{A}_s)\|^2 \mathrm{d}t \quad &\le \int_0^\xi (\mathbf{J}(\overline{\mathbf{A}}_r(t - \tau_r)) - \mathbf{J}(\overline{\mathbf{A}}_s(t - \tau_s)), \mathbf{A}_r(t) - \mathbf{A}_s(t)) \mathrm{d}t \\
+ \nu_M \int_0^\xi (\nabla \times (\mathbf{A}_r - \overline{\mathbf{A}}_r), \nabla \times (\mathbf{A}_r - \mathbf{A}_s)) \mathrm{d}t &+ \nu_M \int_0^\xi (\nabla \times (\mathbf{A}_s - \overline{\mathbf{A}}_s), \nabla \times (\mathbf{A}_r - \mathbf{A}_s)) \mathrm{d}t.
\end{aligned}$$

From the Cauchy–Schwarz and Young's inequalities and the Lipschitz continuity of $\mathbf{J}$, we find

$$\|\mathbf{A}_r(\xi) - \mathbf{A}_s(\xi)\|^2 + \int_0^\xi \|\nabla \times \mathbf{A}_r - \nabla \times \mathbf{A}_s\|^2 \mathrm{d}t \le C_\varepsilon \int_0^\xi \|\mathbf{A}_r - \mathbf{A}_s\|^2 \mathrm{d}t$$
$$+ \varepsilon \int_0^\xi \|\overline{\mathbf{A}}_r(t-\tau_r) - \overline{\mathbf{A}}_s(t-\tau_s)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t + C(\tau_r + \tau_s).$$

For the last term we used Lemma 5. The argument in the $\mathbf{H}(\mathbf{curl}, \Omega)$-norm can be written as $\overline{\mathbf{A}}_r(t-\tau_r) - \overline{\mathbf{A}}_r(t) + \overline{\mathbf{A}}_r(t) - \mathbf{A}_r(t) + \mathbf{A}_r(t) - \mathbf{A}_s(t) + \mathbf{A}_s(t) - \overline{\mathbf{A}}_s(t) + \overline{\mathbf{A}}_s(t) - \overline{\mathbf{A}}_s(t-\tau_s)$. Based on Lemma 3 one can check that $\int_0^\xi \|\overline{\mathbf{A}}(t-\tau) - \overline{\mathbf{A}}(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t \le C\tau$. The last inequality can thus be simplified to

$$\|\mathbf{A}_r(\xi) - \mathbf{A}_s(\xi)\|^2 + \int_0^\xi \|\nabla \times (\mathbf{A}_r - \mathbf{A}_s)\|^2 \le C(\tau_r + \tau_s) + C \int_0^\xi \|\mathbf{A}_r(t) - \mathbf{A}_s(t)\|^2.$$

From the Gronwall lemma we obtain

$$\max_{t\in[0,T]} \|\mathbf{A}_r(t) - \mathbf{A}_s(t)\|^2 + \int_0^T \|\nabla \times (\mathbf{A}_r - \mathbf{A}_s)\|^2 \mathrm{d}t \le C\left(\frac{1}{r} + \frac{1}{s}\right). \tag{14}$$

This result states that the sequence of functions $\mathbf{A}_n$ is a Cauchy sequence in the complete spaces $C\big([0,T], (L^2(\Omega))^3\big)$ and in $L^2((0,T), \mathbf{H}(\mathbf{curl}, \Omega))$. □

From the previous theorem we obtain a $\sqrt{\tau}$-convergence rate. If we assume stronger regularity on the initial condition, i.e., $\nu\nabla \times \mathbf{A}_0 \in \mathbf{H}(\mathbf{curl}, \Omega)$, faster convergence (proportional to τ) is obtained. This is caused by the stronger estimate in Lemma 4.

5. Existence and uniqueness

At this point we can prove the existence of the weak solution of (7). In relation (13) we pass the limit to infinity and prove that the limit function satisfies (7).

Theorem 7 (*Existence*). *Let $\mathbf{A} \in C\big([0,T], (L^2(\Omega))^3\big) \cap L^2((0,T), \mathbf{H}(\mathbf{curl}, \Omega))$ be the limit of the sequence $\mathbf{A}_n$ for $n \to \infty$. Then*

- *$\partial_t \mathbf{A}_n(t) \rightharpoonup \partial_t \mathbf{A}(t)$ in $L^2\big([0,T], (L^2(\Omega))^3\big)$*
- *$\mathbf{A}$ is a weak solution of the problem (1).*

Proof.
- From Lemma 3 we know that $\int_0^T \|\partial_t \mathbf{A}_n(t)\|^2 \le C$ i.e. the sequence $\partial_t \mathbf{A}_n$ is bounded in the Hilbert space $L^2([0,T], (L^2(\Omega))^3)$. Consequently, there exists a weakly convergent subsequence, which we still denote by $\partial_t \mathbf{A}_n$. If $\mathbf{w}(t)$ is the weak limit of this sequence, then we obtain the following diagram of convergences

$$\begin{array}{ccc} \displaystyle\int_0^t (\partial_t \mathbf{A}_n, \boldsymbol{\varphi}) \mathrm{d}t & \to & \displaystyle\int_0^t (\mathbf{w}, \boldsymbol{\varphi}) \mathrm{d}t, \\ \| & & \\ (\mathbf{A}_n(t), \boldsymbol{\varphi}) - (\mathbf{A}_0, \boldsymbol{\varphi}) & \to & (\mathbf{A}(t), \boldsymbol{\varphi}) - (\mathbf{A}_0, \boldsymbol{\varphi}). \end{array}$$

  We conclude that $\mathbf{w} = \partial_t \mathbf{A}(t)$.
- We integrate the identity (13) in time and obtain

$$\int_0^t (\sigma \partial_t \mathbf{A}_n(t), \boldsymbol{\varphi}) \mathrm{d}t + \int_0^t (\nu \nabla \times \overline{\mathbf{A}}_n(t), \nabla \times \boldsymbol{\varphi}) \mathrm{d}t = \int_0^t (\mathbf{J}(\overline{\mathbf{A}}_n(t-\tau)), \boldsymbol{\varphi}) \mathrm{d}t. \tag{15}$$

  For the two terms on the left-hand side we apply the weak convergence of $\partial_t \mathbf{A}_n$ in $L^2\big((0,T), (L^2(\Omega))^3\big)$ and the strong convergence of $\overline{\mathbf{A}}_n$ in $L^2((0,T), \mathbf{H}(\mathbf{curl}, \Omega))$. Convergence of the RHS follows immediately from the strong convergence of $\overline{\mathbf{A}}_n$ and Lemma 3. Taking $n \to \infty$ in relation (15) results in

$$\int_0^t (\sigma \partial_t \mathbf{A}(t), \boldsymbol{\varphi}) + \int_0^t (\nu \nabla \times \mathbf{A}(t), \nabla \times \boldsymbol{\varphi}) \mathrm{d}t = \int_0^t (\mathbf{J}(\mathbf{A}(t)), \boldsymbol{\varphi}) \mathrm{d}t, \quad \forall t \in [0,T].$$

  After time derivation we obtain the weak formulation (7) of our problem. This proves that the limit $\mathbf{A}(t)$ of the sequences of Rothe functions is the unique weak solution to the problem (1). □

From Theorem 6 we immediately get the error estimates for the semi-discrete approximation (13) by taking the limit $r \to \infty$ in (14).

Corollary 1 (*Error Estimate*). *Let the assumption on $\mathbf{A}_0$ in Lemma 3 be satisfied. Then there exists a constant C such that*

$$\max_{t\in[0,T]} \|\mathbf{A}_n(t)-\mathbf{A}(t)\|^2 + \int_0^T \|\nabla\times(\mathbf{A}_n(t)-\mathbf{A}(t))\|^2 \mathrm{d}t \le C\tau.$$

Under the conditions in Lemma 4 we obtain a better estimate

$$\max_{t\in[0,T]} \|\mathbf{A}_n(t)-\mathbf{A}(t)\|^2 + \int_0^T \|\nabla\times(\mathbf{A}_n(t)-\mathbf{A}(t))\|^2 \mathrm{d}t \le C\tau^2.$$

These estimates could also be obtained by subtracting (13) from (7) and using similar steps as in the proof of Theorem 6. From this we can also conclude the uniqueness of the weak solution.

6. Fully discrete finite element approach

We are now ready to study the fully discretized approximation of (7). We will follow a similar approach as [4]. We consider a regular triangulation $\mathcal{T}^h$ of the space domain Ω with tetrahedra. The mesh parameter h is defined in the usual way as the maximum diameter of all tetrahedra. Since we are looking for solutions in the space $\mathbf{H}(\mathbf{curl}, \Omega)$, we will apply curl-conforming edge elements on the mesh. For the second family[1] of curl-conforming elements [7], the following finite element space is proposed

$$V_0^h = \left\{ \mathbf{v}_h \in \mathbf{H}(\mathbf{curl}, \Omega) \;\middle|\; \mathbf{v}_{h|K} \in (\mathcal{P}_1)^3, \forall K \in \mathcal{T}^h \text{ and } \mathbf{v}_h \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \right\}, \tag{16}$$

where $\mathcal{P}_1$ is the space of linear polynomials. On every tetrahedron, 12 degrees of freedom are imposed as the moments along every edge e,

$$M_e(\mathbf{v}) = \left\{ \int_e (\mathbf{v}\cdot\mathbf{t})\, q \,\mathrm{d}s\;, \forall q \in \mathcal{P}_1(e) \right\},$$

which uniquely determine every element of V_0^h. The interpolation operator is only defined on a subspace of $\mathbf{H}(\mathbf{curl}, \Omega)$, since the $\mathbf{H}(\mathbf{curl}, \Omega)$-regularity does not insure that the moments (16) are defined [8]. The interpolation operator Π_h is defined for every $\mathbf{v} \in \mathbf{H}^\alpha(\mathbf{curl}, \Omega)$, $\alpha > 1/2$, such that $\Pi_h\mathbf{v} \in V^h$ has the same degrees of freedom as $\mathbf{v}$. The approximation properties of this operator can be found in [4].

After time and space discretization we obtain the following approximation of our problem

$$(\sigma\delta\mathbf{A}_i^h, \boldsymbol{\varphi}^h) + (\nu\nabla\times\mathbf{A}_i^h, \nabla\times\boldsymbol{\varphi}^h) = (J(\mathbf{A}_{i-1}^h), \boldsymbol{\varphi}^h), \quad \forall\boldsymbol{\varphi}^h \in V_0^h,$$

with $\mathbf{A}_0^h = \Pi_h\mathbf{A}_0$. Existence and uniqueness is again obtained from the Lax–Milgram lemma and we can construct the Rothe functions $\mathbf{A}_n^h(t)$ and $\overline{\mathbf{A}}_n^h(t)$ in the same way as for the semi-discrete case. For the fields $\mathbf{A}_i$, similar estimates as in Lemmas 2 and 3 are valid. We remark that $\nabla\times\mathbf{A}_0^h$ is a piecewise constant function and thus the condition in Lemma 4 can never be satisfied for first order elements. The fully discrete equation in terms of the space discretized Rothe functions is

$$(\sigma\partial_t\mathbf{A}_n^h, \boldsymbol{\varphi}^h) + (\nu\nabla\times\overline{\mathbf{A}}_n^h, \nabla\times\boldsymbol{\varphi}^h) = (\mathbf{J}(\overline{\mathbf{A}}_n^h(t-\tau)), \boldsymbol{\varphi}^h), \quad \forall\boldsymbol{\varphi}^h \in V_0^h. \tag{17}$$

If we subtract (17) from (7), set $\boldsymbol{\varphi}^h = \Pi_h\mathbf{A} - \mathbf{A}_n^h$ and integrate in time, we obtain after some rearrangements

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{A}(t)-\mathbf{A}_n^h(t)\|^2 - \frac{1}{2}\|\mathbf{A}_0-\Pi_h\mathbf{A}_0\|^2 + \int_0^t \|\nabla\times(\mathbf{A}-\overline{\mathbf{A}}_n^h)\|^2\mathrm{d}t \\
&\quad = \int_0^t (\sigma\partial_t(\mathbf{A}-\mathbf{A}_n^h), \mathbf{A}-\Pi_h\mathbf{A})\mathrm{d}t + \int_0^t (\nu\nabla\times(\mathbf{A}-\overline{\mathbf{A}}_n^h), \nabla\times(\mathbf{A}-\Pi_h\mathbf{A}))\mathrm{d}t \\
&\qquad + \int_0^t (\mathbf{J}(\mathbf{A})-\mathbf{J}(\overline{\mathbf{A}}_n^h(t-\tau)), \mathbf{A}-\mathbf{A}_n^h) + \int_0^t (\mathbf{J}(\mathbf{A})-\mathbf{J}(\overline{\mathbf{A}}_n^h(t-\tau)), \Pi_h\mathbf{A}-\mathbf{A}) \\
&\qquad + \int_0^t (\nu\nabla\times(\mathbf{A}-\overline{\mathbf{A}}_n^h), \nabla\times(\mathbf{A}_n^h-\overline{\mathbf{A}}_n^h))\mathrm{d}t := \sum_{i=1}^5 S_i.
\end{aligned}$$

[1] We use the second family, because for a given order this family has better approximating properties in $L^2(\Omega)$ compared to the first family [6]. This comes however with a computational cost, since more degrees of freedom have to be calculated.

The first term S_1 can be bounded after integration by parts, as

$$S_1 \le \varepsilon \|\mathbf{A}(t) - \mathbf{A}_n^h(t)\|^2 + C_\varepsilon \|\mathbf{A}(t) - \Pi_h \mathbf{A}(t)\|^2 + C\|\mathbf{A}_0 - \Pi_h \mathbf{A}_0\|^2 + C\int_0^t \|\mathbf{A} - \mathbf{A}_n^h\|^2 \mathrm{d}t + C\int_0^t \|\partial_t(\mathbf{A} - \Pi_h \mathbf{A})\|^2 \mathrm{d}t.$$

If we assume $\mathbf{A} \in H^1\left([0,T], \mathbf{H}^1(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\mathbf{curl}, \Omega)\right)$, we know that [4]

$$\|\mathbf{A} - \Pi_h \mathbf{A}\|^2 \le Ch^2 \|\mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)},$$

$$\int_0^T \|\partial_t(\mathbf{A} - \Pi_h \mathbf{A})\|^2 \mathrm{d}t \le Ch^2 \int_0^T \|\partial_t \mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} \mathrm{d}t.$$

If $\mathbf{A}_0 \in \mathbf{H}^1(\mathbf{curl}, \Omega)$, $\|\mathbf{A}_0 - \Pi_h \mathbf{A}_0\|^2 \le Ch^2 \|\mathbf{A}_0\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)}$.

For the second term we find [4, Lemma 3.3]

$$S_2 \le \varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \overline{\mathbf{A}}_n^h)\|^2 \mathrm{d}t + C_\varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \Pi_h \mathbf{A})\|^2 \mathrm{d}t,$$
$$\le \varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \overline{\mathbf{A}}_n^h)\|^2 \mathrm{d}t + Ch^2 \int_0^T \|\nabla \times \mathbf{A}\|^2_{H^1(\Omega)} \mathrm{d}t.$$

The third term reduces to

$$S_3 \le C_\varepsilon \int_0^t \|\mathbf{A} - \mathbf{A}_n^h\|^2 \mathrm{d}t + \varepsilon \int_0^t \|\mathbf{A} - \overline{\mathbf{A}}_n^h(t-\tau)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t,$$
$$\le C_\varepsilon \int_0^t \|\mathbf{A} - \mathbf{A}_n^h\|^2 \mathrm{d}t + \varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \mathbf{A}_n^h)\|^2 \mathrm{d}t + C\tau^2 \int_0^T \|\partial_t \mathbf{A}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t.$$

For the forth term we obtain

$$S_4 \le \varepsilon \int_0^t \|\mathbf{A} - \overline{\mathbf{A}}_n^h(t-\tau)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t + C_\varepsilon h^2 \int_0^t \|\mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} \mathrm{d}t,$$
$$\le \varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \mathbf{A}_n^h)\|^2 \mathrm{d}t + C\tau^2 \int_0^T \|\partial_t \mathbf{A}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t + C_\varepsilon h^2 \int_0^T \|\mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} \mathrm{d}t.$$

Finally, the fifth term is bounded by

$$S_5 \le \varepsilon \int_0^t \|\nabla \times (\mathbf{A} - \overline{\mathbf{A}}_n^h)\|^2 \mathrm{d}t + C_\varepsilon \tau^2 \int_0^t \|\partial_t \mathbf{A}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t.$$

Now, we add everything together. We fix ε sufficiently small and apply the Gronwall lemma. If we define the following a priori bound on $\mathbf{A}$ and $\mathbf{A}_0$,

$$m(\mathbf{A}, \mathbf{A}_0) = \max_{t\in[0,T]} \|\mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} + \int_0^T \left(\|\partial_t \mathbf{A}\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)} + \|\nabla \times \mathbf{A}\|^2_{H^1(\Omega)} \right) \mathrm{d}t + \|\mathbf{A}_0\|^2_{\mathbf{H}^1(\mathbf{curl},\Omega)},$$

we obtain the following estimate

$$\max_{t\in[0,T]} \|\mathbf{A}(t) - \mathbf{A}_n^h(t)\|^2 + \int_0^T \|\nabla \times (\mathbf{A} - \overline{\mathbf{A}}_n^h)\|^2 \mathrm{d}t \le m(\mathbf{A}, \mathbf{A}_0) h^2 + C_1 \int_0^T \|\mathbf{A}_n^h - \overline{\mathbf{A}}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t + C_2 \tau^2 \int_0^T \|\partial_t \mathbf{A}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \mathrm{d}t.$$

Theorem 8. *Let the weak solution* $\mathbf{A}$ *and initial condition* $\mathbf{A}_0$ *of problem* (1) *satisfy*

$$\mathbf{A} \in H^1\left([0,T], \mathbf{H}^1(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\mathbf{curl}, \Omega)\right), \qquad \mathbf{A}_0 \in \mathbf{H}^1(\mathbf{curl}, \Omega),$$

then we obtain the following error estimates for the fully discretized scheme

$$\max_{t\in[0,T]} \|\mathbf{A}(t) - \mathbf{A}_n^h(t)\|^2 + \int_0^T \|\nabla \times (\mathbf{A} - \overline{\mathbf{A}}_n^h)\|^2 \mathrm{d}t \le C(h^2 + \tau).$$

If the initial condition satisfies $\nu \nabla \times \mathbf{A}_0 \in \mathbf{H}^1(\mathbf{curl}, \Omega)$*, we obtain a* $C(h^2 + \tau^2)$ *estimate.*

Proof. In the same way as in Lemma 5, we can bound $\int_0^T \|\mathbf{A}_n^h - \overline{\mathbf{A}}_n^h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}$ by $C\tau$. If $\nu \nabla \times \mathbf{A}_0 \in \mathbf{H}^1(\mathbf{curl}, \Omega)$, we can define the compatibility condition $\delta \mathbf{A}_0^h = \mathbf{J}(\mathbf{A}_0^h) - \nabla \times \Pi_h(\nu \nabla \times \mathbf{A}_0)$ and can bound the previous expression by $C\tau^2$. □

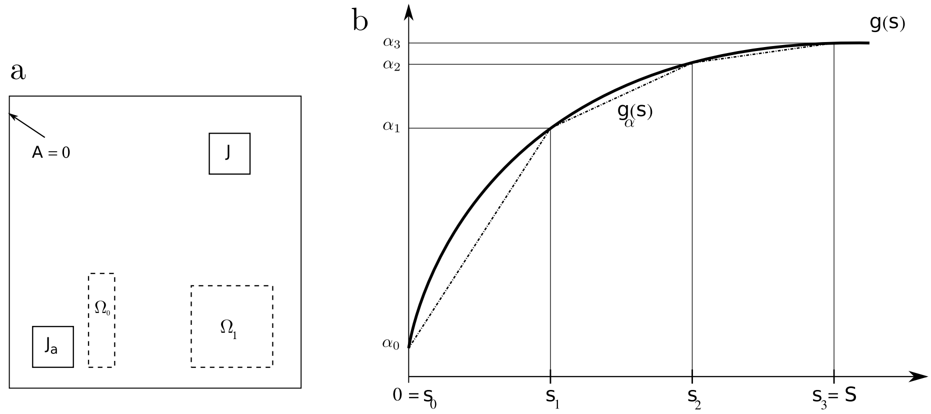

Fig. 1. (a) A scheme of our numerical setup. A current source J_a produces an electromagnetic field which has to be minimized in the target region Ω_1. Therefore a compensation current J is imposed with the amplitude depending on the strength of the field in the region Ω_0. (b) The function $g(s)$ is approximated by the piecewise linear function $g_{\boldsymbol{\alpha}}$. The values α_i are the optimization parameters.

7. Application to magnetic shielding

The model (1) occurs in the optimization problem of magnetic shielding in low-frequency electromagnetism, where a magnetic stray field is minimized in a defined area [9]. A passive and active shield can be constructed near the electromagnetic device, consisting of a conducting wall and a number of compensation coils which generate an opposite field [10]. An interesting problem occurs when an extra winding, positioned close to the excitation coil, is used to drive the compensation coils. This compensation coil is inductively coupled to the excitation coil and works as the secondary winding of a transformer. This was studied in [11] for an axisymmetric induction heater.

Here, we will model this problem for a 2D induction heater, by using a non-local source term, which depends on the magnetic field in a certain area near the excitation current. We allow for a non-linear coupling between source and field, which forces us to work in time domain. The problem is then to reconstruct this non-linear function, such that the field in a certain target area is minimized. A more general approach consists of the reconstruction of this function, such that the potential in a target area equals a certain measured field $\mathbf{A}^*$. The inverse problem is formulated as follows.

Problem 1. Consider the 2D problem

$$\sigma \partial_t A - \nabla \cdot (\nu \nabla A) = J_a + g(\|\mathbf{B}\|_{\Omega_0}) J, \quad \text{in } [0, T] \times \Omega \tag{18}$$

$$A = 0, \quad \text{on } [0, T] \times \partial\Omega \tag{19}$$

$$A = A_0, \quad \text{in } \Omega, \tag{20}$$

where J_a is the applied current in the excitation coil, J is a normalized compensation current and g is a non-linear function of the norm of the magnetic induction $\mathbf{B}$ in a certain domain Ω_0 close to the excitation coil. Our problem is then to find the function $g(s)$, such that the magnetic potential A equals the measured field A^* in the target area Ω_1 i.e. such that the functional

$$F(g) = \frac{1}{2} \int_0^T \|A(g) - A^*\|^2_{\Omega_1} \mathrm{d}t, \tag{21}$$

is minimized.

The problem setup is depicted in Fig. 1. We solve this problem numerically by approximating the function $g(s)$ in the interval $[0, S]$ by a piecewise continuous function. Therefore, the interval $[0, S]$ is divided in a partition $0 = s_0 < s_1 < \cdots < s_n$ and the function value at these points is denoted by α_i, $i = 0, \ldots, n$. Between two points the function is linearly interpolated. We denote the obtained approximation as $g_{\boldsymbol{\alpha}}(s)$. The maximum value S depends on the current sources and the initial value A_0. It should be chosen larger than $\max_{t,x} \|\mathbf{B}(t, x)\|_{\Omega_0}$, $t \in [0, T]$, $x \in \Omega$.

Under these consideration, we obtain the following approximation of our problem, written in variational form.

Problem 2. Find $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$, such that the functional

$$F(\boldsymbol{\alpha}) = \frac{1}{2} \int_0^T \|A(\boldsymbol{\alpha}) - A^*\|^2_{\Omega_1} \mathrm{d}t, \tag{22}$$

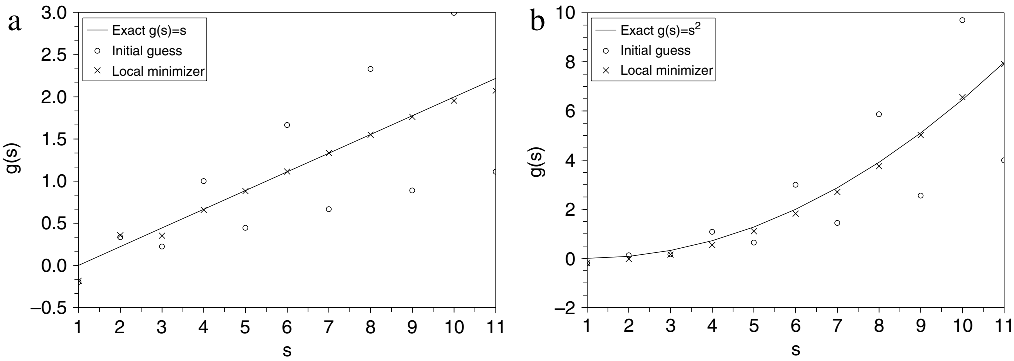

Fig. 2. Numerical results of the reconstruction of a linear (a) and a quadratic (b) current source. We obtain only local convergence of the numerical method, since the cost functional F is not convex.

obtains a minimum in $\boldsymbol{\alpha}$, where $A(\boldsymbol{\alpha})$ is the solution of the problem

$$(\sigma\partial_t A, \phi) + (\nu\nabla A, \nabla\phi) = (J_a, \phi) + g_{\boldsymbol{\alpha}}(\|\mathbf{B}\|_{\Omega_0})(J, \phi), \quad \text{in } [0, T] \times \Omega \tag{23}$$

$$A(0) = A_0, \quad \text{in } \Omega, \tag{24}$$

We will use a gradient optimization procedure to minimize the functional F. If we denote the Gâteaux derivative of F in the point $\boldsymbol{\alpha}$ and direction $\boldsymbol{\mu}$ as $\delta F(\boldsymbol{\alpha}, \boldsymbol{\mu})$, we receive from (22)

$$\delta F(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \lim_{t\to 0} \frac{F(\boldsymbol{\alpha} + t\boldsymbol{\mu}) - F(\boldsymbol{\alpha})}{t} = \int_0^T \big(A(\boldsymbol{\alpha}) - A^*, \delta A(\boldsymbol{\alpha}, \boldsymbol{\mu})\big)_{\Omega_1} \mathrm{d}t.$$

In order to get the Gâteaux derivative of A, we need to solve the sensitivity equation which can be found after formal derivation of Eq. (23) to $\boldsymbol{\alpha}$

$$(\sigma\partial_t \delta A, \phi) + (\nu\nabla\delta A, \nabla\phi) - g'_{\boldsymbol{\alpha}}(\|\nabla A\|_{\Omega_0}) \frac{(\nabla A, \nabla\delta A)_{\Omega_0}}{\|\nabla A\|_{\Omega_0}} (J, \phi) = g_{\boldsymbol{\mu}}(\|\nabla A\|_{\Omega_0})(J, \phi). \tag{25}$$

The components of the gradient of A with respect to $\boldsymbol{\alpha}$ are then obtained by solving the sensitivity equation for $\boldsymbol{\mu}_j = \delta_{ij}$ for $j = 0, \dots, n$. In every iteration step of the minimization procedure $n + 1$ PDEs have to be solved to calculate the gradient. It requires a large computational time and can be avoided by introducing an adjoint problem. This problem is obtained by switching the time derivative on the RHS of (25) to the test function using integration by parts. We now introduce the adjoint variable ξ as the solution of the following problem.

Problem 3 (*Adjoint Equation*). Find ξ such that

$$-(\sigma\partial_t\xi, \phi) + (\nu\nabla\xi, \nabla\phi) - g'_{\boldsymbol{\alpha}}(\|\nabla A\|_{\Omega_0}) \frac{(\nabla A, \nabla\phi)_{\Omega_0}}{\|\nabla A\|_{\Omega_0}} (\xi, J) = (A - A^*, \phi)_{\Omega_1},$$

$$\xi(T) = 0.$$

From the adjoint equation, one immediately obtains the following expression for the Gâteaux derivative of F

$$\delta F(\boldsymbol{\alpha}, \boldsymbol{\mu}) = (\xi, J) \int_0^T g_{\boldsymbol{\mu}}(\|\nabla A\|_{\Omega_0}) \mathrm{d}t.$$

The result is that the full gradient of F can be calculated as $n + 1$ scalar products. We only need to solve an extra PDE for the adjoint variable ξ. Compared to the solution of the sensitivity equation, this means a large reduction of computational time.

The numerical minimization of the functional F is based on the BFGS-method, where gradients are calculated using the adjoint system. All PDEs are solved using the Rothe method and a 2D finite element mesh. From the previous analysis we know that for every set $\boldsymbol{\alpha}$, there is a unique potential A which is well approximated by the fully discretized solution.

In Fig. 2 numerical results are presented for reconstruction of a linear and quadratic current source. In the figure, the initial guess for the iterative minimization procedure (BFGS) is presented together with the local minimizer, which is obtained after less than 150 iterations. Since the cost functional F is not convex, we can only obtain convergence to a local minimum. Therefore, the initial guess has to be sufficiently close to the actual minimizer of the functional.

8. Conclusions

We have studied a non-linear evolution equation for quasi-static electromagnetic fields with a non-local field-dependent source. Under weak conditions on this source and the initial condition, we obtained convergence of our numerical scheme and we presented the corresponding error estimates. Finally, we evaluated our method for the 2D problem of inverse magnetic shielding.

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